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NECKING IN THE TENSILE TEST. CORRECTION FORMULAE AND RELATED ERROR ESTIMATION

PRZEWĘŻENIE W PRÓBIE ROZCIĄGANIA. OSZACOWANIE BŁĘDÓW ZE WZORÓW KORYGUJĄCYCH

This paper deals with analytical modelling of the classical tensile test which is still considered as one of the main experimental procedures to determine the flow curve of elasto-plastic materials. Together with numerical simulation of the deformation process, it allows us to estimate the accuracy of the well-known classical formulae when they are used to evaluate the experimental data under stage of the neck formation. For this aim, errors related with the application of each individual simplifying assumption on the value of the average normalised axial stress were analysed. From critical analysis of one of the assumptions (curvature radius formulae of the longitudinal stress trajectory), a new empirical formula has been derived. It depends on the same measured parameters, however, leads to higher accuracy than any of the classical formulae. In addition, a new formula obtained from analytical investigation, i.e. in Lagrange approach, is discussed.

Keywords: tensile test, material properties, flow curve, yield stress, analytical analysis, FEM simulation, Bridgman formula, Davidenkov-Spiridonova formula

Praca dotyczy analitycznego modelowania klasycznej próby rozciągania, która jest wciąż uważana za jedną z podstawowych procedur eksperymentalnych do wyznaczenia naprężenia uplastyczniającego materiałów sprężysto-plastycznych. Wraz z symulacją numeryczną procesu pozwoliło nam oszacować dokładność znanych wzorów klasycznych podczas wykorzystywania ich do skorygowania danych eksperymentalnych od momentu pojawienia się szyjki w próbce. W tym celu przeanalizowano błędy związane z zastosowaniem każdego założenia upraszczającego na wartość średniego znormalizowanego naprężenia osiowego. Podczas krytycznej analizy założenia o postaci wzoru na promień krzywizny trajektorii naprężenia wzdłużnego otrzymano nowy wzór empiryczny zależny od tego samego parametru, co wzory klasyczne, ale wykazujący większą od nich dokładność. W pracy przedstawiono także nowy wzór otrzymany z analitycznego modelowania zagadnienia w opisie Lagrange'a.

1. Introduction

Tensile testing with axisymmetric specimens is a simple and an important standard engineering procedure which is effective to determine important elastic and plastic properties of materials. Up to the stage of neck formation, it is characterised, in fact, by a homogeneous 1-D stress state that provides a unique opportunity to evaluate the mechanical properties of the specimen material. Since the well-known papers of B r i d g m a n [2] and D a v i d e n k o v - S p i r i d o n o v a [5] have been published, researchers have been applying these simple formulae to extend the knowledge on the yield stress to a wider region when the plastic localisation takes place with pronounced neck formation (see e.g. [1, 3, 4]).

B r i d g m a n and D a v i d e n k o v - S p i r i d o n o v a derived their formulae in frames of the deformation theory of plasticity (or in frames the plastic flow theory) in Euler's coordinates under Huber-Mises or Tresca yield criteria. Accurate mathematical formulation of the respective boundary problems can be found in H i l l ' s monograph [10]. The following assumptions have been additionally employed in the minimum cross-section: a) neglecting of the elastic properties at the stage of the neck creation and resulting material incompressibility within the plastic region, b) equality of the circumferential stress to the radial stress, c) constancy of the yield stress, k, at every time increment, d) utilisation of specific formulae for the radius of cur-

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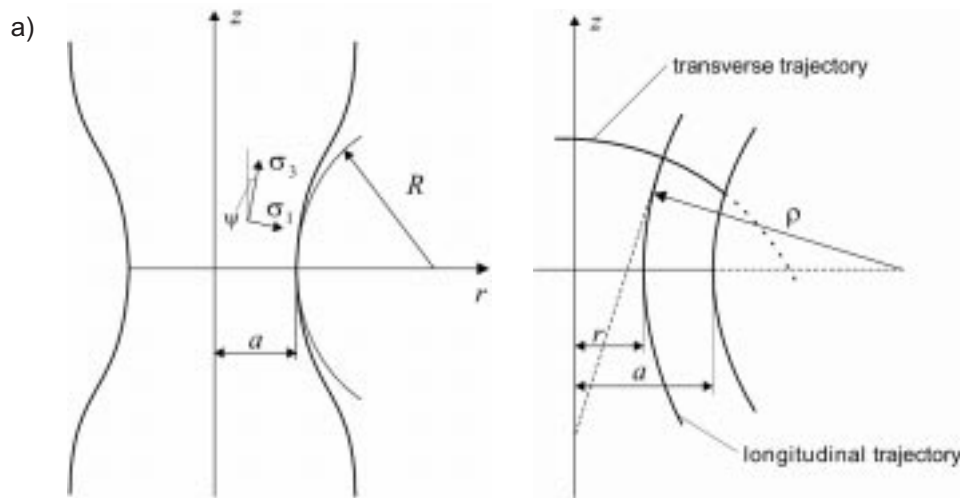
Fig. 1. Neck geometry of a tensile specimen (a); main stress trajectories in the meridian plane (b)

vature of the longitudinal stress trajectory, $\rho = \rho(r)$. Namely, B r i d g m a n [2] derived formulae $(1)_1$ while D a v i d e n k o v - S p i r i d o n o v a [5] assumed $(1)_2$ from experiments:

$$\rho = (a^2 + 2aR - r^2)/(2r), \quad \rho = aR/r, \quad 0 \leqslant r \leqslant a, \quad (1)$$

where R is the external neck radius at point $r = a$ and $z = 0$ (see Fig. 1a).

As a result, the formulae for the normalised axial stress in the minimum cross section were found respectively by B r i d g m a n [2], S i e b e l - D a v i d e n - k o v - S p i r i d o n o v a [5, 16] and S z c z e p i ń s k i [6] in the following forms:

$$\frac{\bar{\sigma}_z}{k} = \left(1 + \frac{2R}{a}\right)\ln\left(1 + \frac{a}{2R}\right), \quad \frac{\bar{\sigma}_z}{k} = 1 + \frac{a}{4R},$$
$$\frac{\bar{\sigma}_z}{k} = \frac{2R}{a}\left[\exp\left(\frac{a}{2R}\right) - 1\right]. \quad (2)$$

Let us note that the same result, even in a more general form in comparison with that obtained by D a v i - d e n k o v - S p i r i d o n o v a, has been earlier derived by S i e b e l and S c h w a i g e r e r [16]. However, their paper was only published in German and is not well known. Also S z c z e p i ń s k i formula $(2)_3$ can hardly be found in engineering literature.

However, from the best author's knowledge any successful attempt to accurately estimate an accuracy of the classical formulae (2) has not been made. Some numerical analyses only confirm that the formulae manifest some errors [4, 12, 13, 15, 17]. Also we would like to refer here to paper [11], where an effort has been made to analytically correct the classic formulae. In [7], we brought the attention to the fact that the error connected with application of B r i d g m a n and D a v i d e n k o v - S p i r i d o n o v a formulae to determine the flow curve can reach even 10% in the case of ideal plastic materials. It is naturally a direct consequence of the assumptions a) – d) used to simplify the derivation of the classical formulae. If the first assumption looks quite reasonable, at least if the neck appears at higher strains, others may be questionable and responsible for the aforementioned error.

In this paper, we concentrate our efforts to estimate the error of formulae (2) connected with the consequent simplifying assumption from b) to d) (assuming at each attempt that the other simplifications are correct). For this purpose, an analytical analysis and appropriate data obtained from numerical simulation are used. Finally, we propose and discuss some new formulae which can be used instead of the classical ones.

2. Applicability of the simplifying assumptions and related error estimations

First, let us describe the numerical model utilised during the error estimation. Due to symmetry, a half of the axisymmetric tensile specimen is considered. The space variables are in the range $z \in (0, 30)$, $r \in (0, 5)$ and the assigned length unit is mm. Dirichlet boundary condition is prescribed at the boundary $z = 30$, linearly changing in z-direction from zero to its maximum value with a small constant increment. Symmetry conditions are applied to the boundary $z = 0$. A very dense mesh, consisting of 64200 elements and refined in the surroundings of the minimum cross-section, was creat-

ed. The material of the considered specimen was assumed to be elasto-plastic with Young's modulus of 210 GPa, Poisson's ratio of 0.3 and the initial yield strength of 200 MPa. We carried out the simulations for three different flow curves modelling a variety of possible material properties. Namely, nonlinear hardening with $k(\bar{\varepsilon}^p) = 100 + 100(1 + 14.24775\bar{\varepsilon}^p)^{0.5}$, linear hardening with a plastic modulus of 150 MPa and, finally, the ideal plasticity model. The numerical simulation was run using MSC. Marc code under an option of the large deformation. The load was implemented with the displacement applied at the end of specimen, so for $z = 30$, in the form $u_z = 4$ mm for ideal plastic material, $u_z = 5$ mm in the case of linear hardening material and $u_z = 12{,}5$ mm for nonlinear hardening material. Taking

into account the specificity of the utilized flow curves, the calculations for ideal plastic material was carried out in 600 steps where every eight step was recorded. In the cases of linear and nonlinear hardening materials we set 200 steps and the recording frequency equals to 10. Computations were verified with respect of both the stability of the obtained solution and the accuracy of the results which was less than 0.1%.

First, we started with the verification of formulae (2) directly using finite element (FE) analysis. The value of the geometrical parameter $\delta = a/R$ was extracted numerically from a post processing analysis of the numerical data. Also the average of the axial and the yield stresses, $\bar{\sigma}_z$ and $\bar{k}$, respectively were calculated from the FEM results. Relevant date is collected in Table 1.

TABLE 1

Relative errors connected with application of particular formulae to determine flow curves for three selected stages of deformation of every material

| Relative error, % | linear hardening | | | nonlinear hardening | | | ideal plasticity | | |
| | Stage of deformation a/R, − | | | Stage of deformation a/R, − | | | Stage of deformation a/R, | | |
	0.371	0.661	1.062	0.134	0.405	0.714	0.591	0.866	1.174
Bridgman	−1.8	−2.8	−5.1	−1.0	−2.5	−4.2	−5.4	−6.7	−8.3
Siebel/Dawid./Spirid	−1.3	−1.5	−2.0	−1.0	−2.0	−2.6	−4.2	−4.5	−4.6
Szczepiński	−0.8	0.2	2.2	−0.9	−1.3	−0.7	−2.8	−1.5	0.6

Only three different levels of plastic deformation (small, medium and large) have been presented. Unfortunately, it was impossible to achieve the same value of the parameter $\delta = a/R$ for different materials under consideration. As a result, we extract values of more or less the same order of magnitude of the parameter δ. As we have mentioned in the introduction, the worst accuracy appears for any of the formulae (2) in the case of the ideal plastic material. This error increases with growing plastic deformation. The first conclusion which can immediately be made from the data presented in Table 1 is that the Bridgman formula (2) is the worst approximation among the classical formulae given in equation (2). This result is quite astonishing since the B r i d g m a n equation is the most frequently used [7] formula.

However, at first glance, it is difficult to conclude which of the simplifying assumptions introduces the greatest error and why. The main goal of the next section is to answer this question. We will sequentially verify each single simplification b) – d), simultaneously assuming that the others are correct. To this aim, the verified assumption is considered in the analytical analysis together with necessary data drawn from the numerical simulation. Finally, we investigate the deviation between obtained and classical solutions. It is the other way of

the error estimation compared to that presented in Table 1, where final results evaluated from classical formulae were compared with those received from the numerical simulation. We are also not going to analyse in that way the Szczepiński formula. The reason is that Szczepiński used the simplified equilibrium equation instead of the accurate equilibrium equation. This brings about an additional error to the theory and must be considered as a further simplifying assumption.

2.1. Assumption of the equality of the circumferential and the radial stresses

First, we consider the assumption b) taking the others, c) and d), for granted. Because the circumferential and radial stresses are not equal to each other in the minimal cross section ($z = 0$), a relation between them can be written in a general form:

$$\sigma_\theta(r) = \sigma_r(r) + \kappa c(r)\sigma_z(r), \quad 0 < r < a, \qquad (3)$$

where κ should be a small parameter and unknown function $c(r)$ is normalised one ($\max |c(r)| = 1$). It is clear that before the neck formation $\sigma_\theta(r) = \sigma_r(r)$ holds ($\kappa = 0$). Thus, the parameter κ can be considered in some

sense as a measure of the plastic localisation connected with the neck formation.

Using (3), H u b e r - M i s e s and T r e s c a yield condition can be rewritten after some algebra in a common form:

$$\sigma_z - \sigma_r = k + \upsilon\kappa c\sigma_z + O(\kappa^2), 0 < \kappa << 1, \qquad (4)$$

where parameter υ takes the values:

$$\upsilon_{HM} = \tfrac{1}{2}, \upsilon_T = 1, \qquad (5)$$

adequately to H u b e r - M i s e s and T r e s c a conditions and $O(\kappa^2)$ as usual denotes the standard Landau symbol utilised to the asymptotical analysis. Differentiating the yield criterion (4) with respect to r, and inserting the result into the equilibrium equation, the following differential equation can be obtained in the minimal cross section:

$$(1 - \upsilon\kappa c)\frac{\partial\sigma_z}{\partial r} - \left(\upsilon\kappa c' + \frac{\kappa c}{r}\right)\sigma_z + \frac{k}{\rho} + \frac{1}{r}O(\kappa^2) = 0,$$
$$0 < \kappa << 1. \qquad (6)$$

The solution of equation (6) together with the boundary condition at the free sample surface ($\sigma_r = 0$) is sought with the regular perturbation technique with respect to the small parameter κ. Indicating the normalised axial stress appeared in the classical formulae by $\tilde{\sigma}_z^{(0)}$, the solution of equation (6) takes form:

$$\frac{\sigma_z}{k} = \tilde{\sigma}_z^{(0)} + \upsilon\kappa c(t)\tilde{\sigma}_z^{(0)}(t) + \kappa\int_0^t \frac{c}{r}\tilde{\sigma}_z^{(0)}dr, \qquad (7)$$

with accuracy of order $O(\kappa^2)$. As a result, the difference between the classical formulae (2) for the average stress and the derived one is:

$$\frac{\Delta\bar{\sigma}_z}{k} = \frac{2\kappa}{a^2}\int_0^a \left(\upsilon c(t)\tilde{\sigma}_z^{(0)}(t) + \int_0^t \frac{c(r)}{r}\tilde{\sigma}_z^{(0)}(r)dr\right)tdt. \quad (8)$$

Knowing the value of the small parameter κ as well as the function $c(r)$ from (8), one can estimate the error of the classical formulae related to this assumption. For this purpose, we extract representative magnitudes of κ and $c(r)$ from the numerical simulation provided for the mentioned three different materials [9]. It has been observed that the value of parameter κ grows for small stages of deformation and takes its maximum value for parameter δ near to 0.2 and then decreases. On the other hand, the function $c(r)$ is increasing with r for almost all stage of deformation. Only for very large plastic strains it changes its behaviour. Integrating (8) with taking advantages of the range of values of the parameter κ and the function $c(r)$ obtained from the numerical simulation, the relative errors of the average normalised axial stress can be received. Because of this procedure, the same value of parameter δ could be utilised for each material under consideration. Appropriate data is collected in Table 2. It is interesting to note that in contrast to the Table 1, where the formulae clearly manifested different accuracy, there is now practically no difference between the formulae (2) in Table 2. However, the error itself is not small.

TABLE 2

Relative error occurred with introduction of the parameter κ for three selected stages of deformation of ideal plastic material

	ideal plasticity		
Relative error, %	Stage of deformation a/R, −		
	8E-05	0.185	1.509
Bridgman	−0.6	−6.7	−5.6
Siebel/Dawid./Spirid	−0.6	−6.7	−5.8

2.2. Assumption of the constancy of yield stress in the surface of minimum cross-section

Let us deal with the next simplifying assumption c), supposing that the rest of them, b) and d), are valid. Let us evolve Taylor's formula for the yield stress in the form:

$$k(\varepsilon_{eq}) = k(\bar{\varepsilon}_{eq}) + k'(\bar{\varepsilon}_{eq})(\varepsilon_{eq} - \bar{\varepsilon}_{eq}) + O(\varepsilon_{eq} - \bar{\varepsilon}_{eq})^2. \quad (9)$$

Taking advantages of the equilibrium equation together with the boundary condition on the neck contour, we have:

$$\sigma_r = \int_r^a \frac{k(\varepsilon_{eq})}{\rho(t)}dt, \qquad (10)$$

where $\varepsilon_{eq} = \varepsilon_{eq}(r)$, and $\bar{\varepsilon}_{eq}$ is its average value across the minimal cross-section. Inserting representation (9) into (10) and taking into account the yield condition

$$\sigma_z - \sigma_r = k(\varepsilon_{eq}), \qquad (11)$$

written (for both yield criteria) in the same form [9] we obtain:

$$\sigma_z - k(\varepsilon_{eq}) = \bar{k}\int_r^a \frac{dt}{\rho(t)} + \bar{\varepsilon}_{eq}k'(\bar{\varepsilon}_{eq})\int_r^a \frac{\Delta\varepsilon_{eq}(t)}{\rho(t)}dt. \quad (12)$$

Here we have introduced the notations $\Delta\varepsilon_{eq} = (\varepsilon_{eq} - \bar\varepsilon_{eq})/\bar\varepsilon_{eq}$ and $\bar{k} = k(\varepsilon_{eq})$. After same transformations, we finally get the increase of the average normalized axial stress in comparison with the classical formulae in the form:

$$\frac{\Delta\bar\sigma_z}{\bar{k}} = \frac{1}{\bar{k}}k'(\bar\varepsilon_{eq})\left[\bar\varepsilon_{eq}\frac{2}{a^2}\int_0^a r\left(\Delta\varepsilon_{eq}(r) + \int_r^a \frac{\Delta\varepsilon_{eq}(t)}{\rho(t)}dt\right)dr\right].$$ (13)

Let us note that the multiplicand on the right-hand side of equation (13) is limited by one for a large majority of the flow curves used in the theory of plasticity and engineering practice. This term takes the biggest value for linear hardening. Making use of representative values of $\Delta\varepsilon_{eq}$ obtained from the numerical simulation, the maximal relative error has been determined for the same value of the parameter δ. Corresponding data is collected in Table 3. Also no difference can be observed between the formulae (2). Moreover, the error is very small and slightly depends on the level of the plastic deformation and does not exceed 0.5%.

TABLE 3

Relative error occurred with taking into account the variability of the yield stress in the minimum section surface of sample for three selected stages of deformation of linear hardening material

Relative error, %	linear hardening		
	Stage of deformation a/R, −		
	0.371	0.661	1.062
Bridgman	−0.4	−0.5	−0.5
Siebel/Dawid./Spirid	−0.4	−0.5	−0.5

It should be noted here that ideal plasticity is characterised by a constant yield stress, so, $k'(\varepsilon_{eq}) \equiv 0$, and such a case has not been naturally considered in this analysis.

2.3. Specific formula for the curvature radius of the longitudinal stress trajectory

Let us now assume that the assumptions b) and c) are valid, but both formulae for the curvature radius ρ (by B r i d g m a n or D a v i d e n k o v and S p i r i d o n o v a) are not good approximation. Note that the distribution of the curvature of the main stress trajectory changes not only with the geometric parameter, r, but, generally speaking, also with the stage of plastic deformation. As a result, one can conclude that $\rho = \rho(r, \bar\varepsilon_{eq})$ and the final formulae for the axial stress becomes

$$\sigma_z = k\left(1 + \int_r^a \frac{dt}{\rho(t, \bar\varepsilon_{eq})}\right), \quad \frac{\bar\sigma_z}{k} = 1 + \frac{1}{a^2}\int_0^a \frac{r^2 dr}{\rho(r, \bar\varepsilon_{eq})},$$ (14)

which depends on the additional incremental parameter $\bar\varepsilon_{eq}$. This brings an additional difficulty to the verification. Namely, we have to extract not only the radius distribution itself, but to simultaneously provide the computations at every time step. The curvature of the radius can be represented in a general form:

$$\rho(r, \bar\varepsilon_{eq}) = Ra\left(rG'(r^2)\right)^{-1},$$ (15)

where properties of the function $G(t)$ were discussed in [7]. Generally speaking, this function is different at each incremental step: $G(t) = G(t, \bar\varepsilon_{eq})$. It was also shown in [7] that for some reasonable values of the parameter, the formula

$$G'(t) = \beta + (a/\sqrt{t})^\alpha(1 - \beta)$$ (16)

can be used to approximate the curvature. Under such an assumption, formula $(14)_2$ take the form:

$$\frac{\bar\sigma_z}{k} = 1 + \frac{a}{4R} + \frac{a(1 - \beta)\alpha}{4R(4 - \alpha)}.$$ (17)

TABLE 4

Relative errors occurred with introduction of $\rho(r)$ determined from the numerical simulation to classical formulae for three selected stages of deformation of every material

Relative error, %	linear hardening			nonlinear hardening			ideal plasticity		
	Stage of deformation a/R, −			Stage of deformation a/R, −			Stage of deformation a/R,		
	0.371	0.661	1.062	0.134	0.405	0.714	0.591	0.866	1.174
Bridgman	0.8	1.5	2.7	0.4	1.0	1.8	1.7	2.5	3.5
Siebel/Dawid./Spirid	0.3	0.1	−0.2	0.3	0.4	0.3	0.6	0.4	0.1

However, the parameters $\alpha, \beta = \alpha, \beta(\varepsilon_{eq})$ can be different at each incremental step. This enables us to estimate the possible error of the classic formulae related to the specific choice of the formula for the curvature radius. The applied procedure is as follows: at each step, the value of the curvature radius is calculated from the FEM computation using 10 points within the interval (0, a). For this reason, we used the definition of the radius based on the stress components and took into account its numerical distribution in the neighbourhood of the minimal cross section. Then we approximated the curve with formula (16) taking care about the accuracy of the approximation. As a result, the values of these two auxiliary parameters, i.e. $\alpha, \beta = \alpha, \beta(\bar{\varepsilon}_{eq})$ were obtained at each time step. Finally, for the particular level of the plastic deformation, the error between the classic formulae and formulae (17) was calculated (cf. Tab. 4).

From the results shown in Tab. 4, it follows that the relative error of the Bridgman formula increases with increasing strain. It should be noted here that the reference value for the error estimation is based on the data extracted from the FE simulation. The maximum error occurs for the ideal plastic material and takes 3.5% for the last stage of deformation. For S i e b e l - D a v i - d e n k o v - S p i r i d o n o v a formula, the error is of the same magnitude (less than 0.6%). In order to easily compare all the obtained results, Tab. 5 collects all the maximum errors calculated during the investigation of each individual simplifying assumption. Furthermore, the material type and the stage of deformation at which these maximum errors occurred are indicated. The error distribution as a function of stage of deformation is symbolised by arrows. For this reason, all tables show error sign, since no norm was considered and only relative errors between the formulae were calculated. One can also observe that the total error of the classical formulae indicated in Table 1 cannot be obtained as a simple sum of the particular errors.

TABLE 5

Maximum errors following from considered simplified assumptions

Simplifying assumptions	Maximum error, %	Behaviour of the error	Material	Stage of deformation a/R, –
specific formula for $\rho(r)$	3.5	↗	ideal plasticity	1.174
$\sigma_\theta = \sigma_r$	−6.7	↘	ideal plasticity	0.185
$k = const$	−0.5	↘	linear hardening	0.661

Summarising above considerations, it should be noticed that each assumption influences the result in a different way. The greatest negative influence on the final result has the assumption of the equality of the circumferential and the radial stresses and the specific form of the curvature radius of the longitudinal stress trajectory. The most reasonable simplification is that the yield stress is constant along the minimum cross-section.

3. Are improvements of the classical formulae possible?

Formula (17) also gives us a chance to derive a new approximating formula for the average normalised axial stress: optimising parameters α, β in equation (17) in order to have the best fit for the different numerical curves which are obtained for different material sets. Figure 2 shows the results obtained from the numerical simulation (marked by dots) and curves evaluated from the new formulae (17) to find the optimal parameter set α, β for different materials. Particularly, for $\alpha = 0.95$ and $\beta = 0.5$, formulae (17) provides a better result than any of the classical formulae. However, it approaches the real curves from both sides at different stages of plastic deformation. If one would like to have a one-side estimation, other values can be recommended: namely, $\alpha = 0.5$ and $\beta = 0.5$, which are not the best fit, but this approximation always lies lower than any curve of the different materials. As suggested in [14], the analysis is restricted to values less than 2 for the parameter a/R.

Additionally, curves corresponding to B r i d g - m a n , S i e b e l - D a v i d e n k o v - S p i r i d o n o - v a and S z c z e p i ń s k i formulae are shown in Fig. 2. As can be seen, the Szczepiński formulae, which is derived in a similar way to the classical ones, reveals a slightly better approximation that the other classical formulae. Formula (17) together with two sets of empirical parameters (α, β) seems to be better than any of the known classical formulae. However, it was created during elimination of only one of the questionable assumptions. As a result, further verification which includes other materials and other computations is required.

Let us mention here that points $\bar{\sigma}_z/\bar{k}$ in Fig. 2 corresponding to the ideal plasticity do not follow a straight line near the origin of the coordinate system. This means that ideas other than the classical ones from papers [2, 5, 6, 16] should probably be employed. Additionally, other parameters may be taken into account to improve the
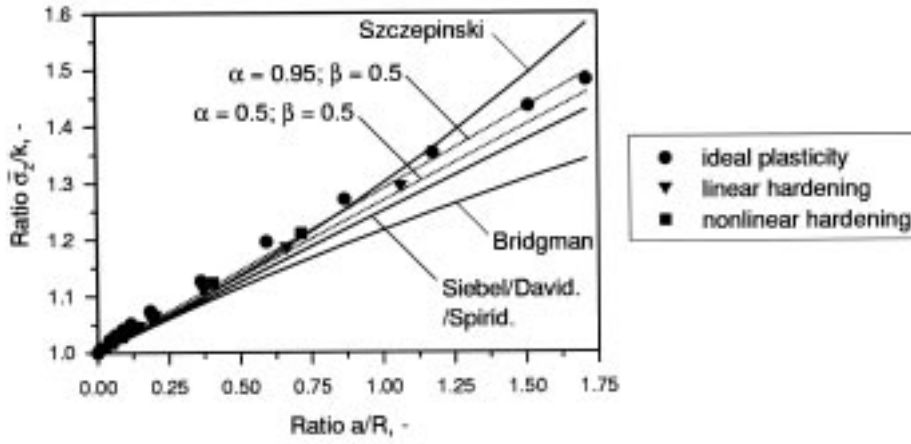
Fig. 2. Ratio $\bar{\sigma}_z/\bar{k}$ as a function of a/R obtained from FEM simulation and its approximations by the classical formulae by Bridgman, Siebel-Davidenkov-Spiridonova and Szczepiński and by the new empirical formula

classical and the empirical formulae (17). In the following, we try to evaluate a new formula based on deformation theory of plasticity under L a g r a n g e ' s coordinate approach [9]. We assume incompressibility of the material in the minimal cross section, but none of the simplifying assumptions like b)-d) is used. Functions describing displacement filed are taken as polygons of a specific degree, while its coefficients are calculated from the natural and symmetry boundary conditions. Finally, only H u b e r - M i s e s yield criterion is applied in the analysis. Trying to achieve a new quantity, we introduced an additional parameter, $\Lambda_0 = 1 - a/a_0$ which is easily measured during an experiment. As a result of the analytical modelling, the formula for the average normalised axial stress was derived in the form [8]:

$$
\begin{aligned}
\frac{\bar{\sigma}_\varsigma}{\bar{k}} &= 1 - \frac{b_{32}a_0^2}{28b_{30}} - \frac{b_{12}a_0^2}{14b_{30}} + \frac{2}{7} + \frac{3b_{10}b_{32}a_0^2}{98b_{30}^2} \\
&+ \frac{3b_{10}}{7b_{30}}\left(\frac{b_{12}a_0^2}{14b_{30}} - \frac{2}{7} - \frac{3b_{10}b_{32}a_0^2}{98b_{30}^2}\right)\ln\left|1 + \frac{7b_{30}}{3b_{10}}\right|.
\end{aligned}
\tag{18}
$$

Unfortunately, from the conditions mentioned above, it was impossible to determine in an unique way the formulae for each coefficients b_{ij} included in formula (18). When we made an additional crucial assumption that $b_{32} = 0$, which can be justified for a small deformation, the accuracy of the new formula was quite good (even better than the classical ones) for small value of parameter a/R [8]. However, it becomes much worse for large deformations. The reason for this was clearly the assumption itself, i.e. $b_{32} = 0$. Additional error probably comes from the fact that we used the linear equilibri-

um equation in our analysis. In this paper, we suggest using other empirical value of the parameter: namely $b_{32} = (2\delta_0\Lambda_0)/a_0^2$, where $\delta_0 = a_0/R$ is different in comparison with the parameter $\delta = a/R$ which appeared in the classical formulae. It is clear that for small deformations, this assumption is consistent with the previous one. Then, making additional use of asymptotic analysis (see [9]), coefficients b_{ij} can be evaluated in form:

$$
b_{10} = \frac{-8\Lambda_0 + \delta_0 - 4\delta_0\Lambda_0}{8}, \quad b_{12} = \delta_0\frac{1 - 2\Lambda_0}{2a_0^2},
$$
$$
b_{30} = \delta_0\frac{1 - 4\Lambda_0}{8}.
\tag{19}
$$

This solution is good enough in the considered parameter range a/R (cf. Fig. 3). The classical formulae are also presented for comparison.

Unfortunately, this new formula behaves worse than the empirical formula. Thus, it is still a demanding task to derive a new formula which is better than the empirical one. One may think about a similar analysis under Euler's approach, where the equilibrium equation is linear.

Summarising our investigations, we can state that the classical formulae can provide a significant error for materials undergoing large plastic deformations with pronounced neck formation, especially in the case of a slightly hardening material. We estimated the range of the possible error and suggested three new formulae which are all better then the classical ones. Empirical formula (17) with specific values of the parameters α, β still requires a serious check for other materials.
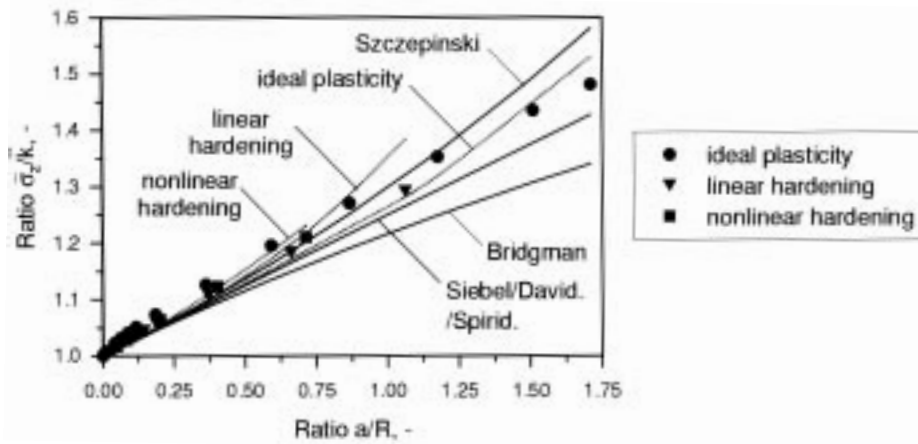
Fig. 3. Ratio $\bar{\sigma}_z/\bar{k}$ as a function of a/R obtained from FEM simulation and its approximations by the classical formulae by Bridgman, Siebel-Davidenkov-Spiridonova and Szczepiński and by the new formula derived in Largange's coordinates

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